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ON THE UNIQUENESS OF THE STRONGLY IRREDUCIBLE DECOMPOSITIONS OF OPERATORS UP TO SIMILARITY

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ABSTRACT. We give a generalization of the Jordan canonical form theorem for a class of bounded linear operators on complex separable Hilbert spaces in terms of direct integrals. Precisely, we study the uniqueness of strongly irreducible decompositions of the operators on the Hilbert spaces up to similarity.

KEYWORDS: *Strongly irreducible operator, von Neumann algebra, K_0 group, direct integral.*

MSC (2000): Primary 47A67; Secondary 47A15, 47C15.

1. INTRODUCTION

Throughout this article, all Hilbert spaces discussed are *complex and separable*. Denote by $\mathscr{L}(\mathscr{H})$ the set of bounded linear operators on a Hilbert space $\mathscr{H}$. An *idempotent* P is an operator in $\mathscr{L}(\mathscr{H})$ satisfying $P^2 = P$. A *projection* Q is an idempotent such that $\ker Q = (\mathrm{ran} Q)^{\perp}$ (See [5]). An operator A in $\mathscr{L}(\mathscr{H})$ is said to be *irreducible* if its commutant $\{A\}' \triangleq \{B \in \mathscr{L}(\mathscr{H}) : AB = BA\}$ contains no projections other than 0 and the identity operator I on $\mathscr{H}$, introduced by P. Halmos in [11]. (The separability assumption is necessary because on a non-separable Hilbert space every operator is reducible.) An operator A in $\mathscr{L}(\mathscr{H})$ is said to be *strongly irreducible* if XAX^{-1} is irreducible for every invertible operator X in $\mathscr{L}(\mathscr{H})$ [10]. This shows that the commutant of a strongly irreducible operator contains no idempotents other than 0 and I. Strong irreducibility stays invariant up to similar equivalence while irreducibility is only an invariant up to unitary equivalence. An idempotent P in $\{A\}'$ is said to be *minimal* if every idempotent Q in $\{A\}' \cap \{P\}'$ satisfies $QP = P$ or $QP = 0$. For a minimal idempotent P in $\{A\}'$, it can be observed that the restriction $A|_{\mathrm{ran}P}$ is strongly irreducible on $\mathrm{ran}P$. An operator A in $\mathscr{L}(\mathscr{H})$ is said to have a finite *strongly irreducible decomposition* if there exist finitely many minimal idempotents $\{P_i\}_{i=1}^n$ in $\{A\}'$ such that $\sum_{i=1}^n P_i = I$ and $P_iP_j = P_jP_i = 0$ for $1 \leq i \neq j \leq n$. By the above observation, an

operator A in $\mathscr{L}(\mathscr{H})$ having a finite strongly irreducible decomposition can be expressed as a direct sum of finitely many strongly irreducible operators.

On finite dimensional Hilbert spaces, every strongly irreducible operator is similar to a Jordan block. In [12], D. A. Herrero and C. Jiang proved that for every operator T in $\mathscr{L}(\mathscr{H})$, there exists a sequence $\{T_n\}_{n=1}^{\infty}$ in $\mathscr{L}(\mathscr{H})$ such that $\lim_{n\to\infty}\|T-T_n\| = 0$, where every operator T_n is similar to a direct sum of finitely many strongly irreducible operators. Y. Cao, J. Fang and C. Jiang [4] studied the uniqueness of finite strongly irreducible decompositions of operators in $\mathscr{L}(\mathscr{H})$ up to similar equivalence by the K_0 groups of Banach algebras. For more work around this subject, the reader is referred to [7, 8, 9, 13, 14, 15, 16, 17, 19]. Inspired by the ideas and results in [4], we study operators in $\mathscr{L}(\mathscr{H})$ which may have no finite strongly irreducible decompositions. In particular, there are many operators in $\mathscr{L}(\mathscr{H})$ whose commutants contain no minimal idempotents. To represent these operators, direct sums of strongly irreducible operators need to be generalized to direct integrals with some regular Borel measures. In [18], C. Jiang and the author of the present paper proved that an operator A in $\mathscr{L}(\mathscr{H})$ is similar to a direct integral of strongly irreducible operators if and only if its commutant $\{A\}'$ contains a bounded maximal abelian set of idempotents. A direct integral of strongly irreducible operators means the integrand is strongly irreducible almost everywhere on the domain of integration. For related concepts and results about direct integrals and abelian von Neumann algebras, the reader is referred to [3, 5, 6, 20, 21].

Following the notation of [18], we generalize a definition mentioned above. An operator A in $\mathscr{L}(\mathscr{H})$ is said to have a *strongly irreducible decomposition* if its commutant $\{A\}'$ contains a bounded maximal abelian set of idempotents. Furthermore, a strongly irreducible decomposition of the operator A is said to be *unique up to similarity* if for bounded maximal abelian sets of idempotents $\mathscr{P}$ and $\mathscr{Q}$ in $\{A\}'$, there is an invertible operator X in $\{A\}'$ such that $X\mathscr{P}X^{-1} = \mathscr{Q}$.

As a corollary of the main theorems, a normal operator in $\mathscr{L}(\mathscr{H})$ has unique strongly irreducible decomposition up to similarity if and only if the multiplicity function m_N for N is finite a. e. on $\sigma(N)$ with respect to the scalar-valued spectral measure μ_N. By this, the tensor product $I_{\mathscr{H}} \otimes N$ does not have unique strongly irreducible decomposition up to similarity, if $\dim\mathscr{H} = \infty$.

To simplify the statements of the main theorems, we need to introduce the upper triangular representation for operator-valued matrices. Assume A in $\mathscr{L}(\mathscr{H})$ is a direct integral of strongly irreducible operators in the form

$$A = \bigoplus_{n=1}^{\infty} \int_{\Lambda_n} A(\lambda) d\mu(\lambda), \tag{1}$$

with respect to a partitioned measure space $\{\Lambda, \mu, \{\Lambda_n\}_{n=1}^{\infty}\}$, where μ is a regular Borel measure on a compact set Λ and $\{\Lambda_n\}_{n=1}^{\infty}$ is a Borel partition of Λ, and the equation $\mu(\Lambda_n) = 0$ holds for all but finitely many n in $\mathbb{N}$ ($0 \notin \mathbb{N}$), and the dimension of the fibre space $\mathscr{H}_\lambda$ ([1], §2) is n for almost every λ in Λ_n.

By ([2], Corollary 2), there is a unitary operator U such that

$$UAU^* = \bigoplus_{n=1}^{\infty} \int_{\Lambda_n} \begin{pmatrix} M_{\phi_n} & M_{\phi_{12}^n} & M_{\phi_{13}^n} & \cdots & M_{\phi_{1n}^n} \\ 0 & M_{\phi_n} & M_{\phi_{23}^n} & \cdots & M_{\phi_{2n}^n} \\ 0 & 0 & M_{\phi_n} & \cdots & M_{\phi_{3n}^n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M_{\phi_n} \end{pmatrix}_{n\times n} (\lambda) d\mu_n(\lambda), \qquad (2)$$

where $\mu_n = \mu|_{\Lambda_n}$ for $1 \leq n < \infty$ and $\phi_n, \phi_{ij}^n \in L^\infty(\mu_n)$, and M_{ϕ_n}, $M_{\phi_{ij}^n}$ are multiplication operators. Denote by $\nu_n = \mu_n \circ \phi_n^{-1}$ the scalar-valued spectral measure for M_{ϕ_n}. Let the set $\{\Gamma_{nm}\}_{m=1}^{m=\infty}$ be the Borel partition of the spectrum $\sigma(M_{\phi_n})$ with respect to the ν_n-measurable multiplicity function m_{ϕ_n} for M_{ϕ_n} defined on $\sigma(M_{\phi_n})$ such that $m_{\phi_n}(\lambda) = m$ for almost every λ in Γ_{nm}. Write ν_{nm} for $\nu_n|_{\Gamma_{nm}}, 1 \leq m \leq \infty$.

For a class of operators in $\mathscr{L}(\mathscr{H})$ having unique strongly irreducible decompositions up to similarity, we give a necessary and sufficient condition by K-theory for Banach algebras. Precisely, we prove the following theorems.

THEOREM 1.1. *Assume that an operator A in $\mathscr{L}(\mathscr{H})$ is stated as in (1) and expressed as in (2) such that*

(i) *the ν_n-measurable multiplicity function m_{ϕ_n} is simple and may take ∞ on the spectrum $\sigma(M_{\phi_n})$ for every n in $\mathbb{N}$ and*
(ii) *every superdiagonal entry as in (2) is invertible for n in $\{n \in \mathbb{N} : \mu(\Lambda_n) > 0\}$.*

Then the following statements are equivalent.

(a) *The strongly irreducible decomposition of A is unique up to similarity.*
(b) *There exists a bounded $\mathbb{N}$-valued simple function r_A on $\sigma(A)$ such that*
$V(\{A\}') \cong \{f(\lambda) \in \mathbb{N}^{(r_A(\lambda))} : f$ *is Borel and bounded on* $\sigma(A)\}$ *and*
$K_0(\{A\}') \cong \{f(\lambda) \in \mathbb{Z}^{(r_A(\lambda))} : f$ *is Borel and bounded on* $\sigma(A)\}$.

THEOREM 1.2. *If an operator A in $\mathscr{L}(\mathscr{H})$ is expressed as in (1) and (2) such that the ν_n-measurable multiplicity function m_{ϕ_n} is simple and bounded on $\sigma(M_{\phi_n})$ for every n in $\mathbb{N}$, then there exists a sequence of operators $\{A_k\}_{k=1}^{\infty}$ in $\mathscr{L}(\mathscr{H})$ required as in Theorem 1.1 and having unique strongly irreducible decompositions up to similarity such that* $\lim_{k\to\infty} \|A_k - A\| = 0$.

2. PROOFS

The following lemma describes an important property of the superdiagonal entries in (2).

LEMMA 2.1. *An upper triangular matrix in $M_n(\mathbb{C})$ of the form*

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1n} \\ 0 & \alpha_{22} & \alpha_{23} & \cdots & \alpha_{2n} \\ 0 & 0 & \alpha_{33} & \cdots & \alpha_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{nn} \end{pmatrix}$$

is strongly irreducible if and only if the equation $\alpha_{11} = \alpha_{22} = \cdots = \alpha_{nn}$ and the inequality $\alpha_{i,i+1} \neq 0$ for $1 \leq i \leq n-1$ both hold.

Proof. If the matrix is strongly irreducible, then the equation $\alpha_{11} = \cdots = \alpha_{nn}$ holds. Write α for α_{ii}, $1 \leq i \leq n$. Because every strongly irreducible matrix is similar to a Jordan matrix, we know that there is an invertible matrix in $M_n(\mathbb{C})$ such that

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1n} \\ 0 & \alpha_{22} & \alpha_{23} & \cdots & \alpha_{2n} \\ 0 & 0 & \alpha_{33} & \cdots & \alpha_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{nn} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \cdots & x_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \cdots & x_{nn} \end{pmatrix} \begin{pmatrix} \alpha & 1 & 0 & \cdots & 0 \\ 0 & \alpha & 1 & \cdots & 0 \\ 0 & 0 & \alpha & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha \end{pmatrix}.$$

This equation yields that $x_{ij} = 0$ for $i > j$ and $x_{ii} = \alpha_{i,i+1} x_{i+1,i+1}$ for $i = 1, 2, \ldots, n-1$. Hence, we obtain $x_{kk} = \prod_{i=k}^{n-1} \alpha_{i,i+1} x_{nn}$. If $\alpha_{i,i+1} = 0$ for some i in $\{1, 2, \ldots, n-1\}$, then the matrix $(x_{ij})_{1 \leq i,j \leq n}$ is not invertible. Therefore the inequality $\alpha_{i,i+1} \neq 0$ holds for $1 \leq i \leq n-1$.

On the other hand, if $\alpha_{i,i+1} \neq 0$ holds for $i = 1, 2, \ldots, n$, then every matrix in $M_n(\mathbb{C})$ commuting with the matrix $(\alpha_{ij})_{1 \leq i,j \leq n}$ can be expressed in the form

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ 0 & x_{11} & x_{23} & \cdots & x_{2n} \\ 0 & 0 & x_{11} & \cdots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_{11} \end{pmatrix}.$$

If X is an idempotent, then it must be I or 0. Thus the matrix $(\alpha_{ij})_{1 \leq i,j \leq n}$ is strongly irreducible. ∎

Applying this lemma, we obtain the following corollary.

COROLLARY 2.2. *In (2), the function $\phi^n_{i,i+1}$ in $L^\infty(\mu_n)$ satisfies $\phi^n_{i,i+1}(\lambda) \neq 0$ almost everywhere on Λ_n for $i = 1, 2, \ldots, n-1$.*

In this corollary, the Multiplication operator $M_{\phi^n_{i,i+1}}$ induced by the function $\phi^n_{i,i+1}$ is not invertible in general. But $M_{\phi^n_{i,i+1}}$ can be approximated by a sequence of invertible Multiplication operators in $\mathscr{L}(L^2(\mu_n))$. Meanwhile, replacing the superdiagonal entries with invertible ones enable us to simplify the problem. That is why we add the hypothesis (ii) in Theorem 1.1. Precisely, we obtain the following two lemmas.

LEMMA 2.3. *If an operator A_n is a direct integral of strongly irreducible operators stated as in (2) in the form*

$$A_n = \begin{pmatrix} M_{\phi_n} & M_{\phi^n_{12}} & M_{\phi^n_{13}} & \cdots & M_{\phi^n_{1n}} \\ 0 & M_{\phi_n} & M_{\phi^n_{23}} & \cdots & M_{\phi^n_{2n}} \\ 0 & 0 & M_{\phi_n} & \cdots & M_{\phi^n_{3n}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M_{\phi_n} \end{pmatrix}_{n\times n},$$

then for every positive integer k, there exists an operator A_{nk} in the form

$$A_{nk} = \begin{pmatrix} M_{\phi_n} & M_{\phi^n_{12,k}} & M_{\phi^n_{13}} & \cdots & M_{\phi^n_{1n}} \\ 0 & M_{\phi_n} & M_{\phi^n_{23,k}} & \cdots & M_{\phi^n_{2n}} \\ 0 & 0 & M_{\phi_n} & \cdots & M_{\phi^n_{3n}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M_{\phi_n} \end{pmatrix}_{n\times n}$$

with invertible $M_{\phi^n_{i,i+1,k}}$ for $1 \leq i \leq n-1$ such that $\|A_n - A_{nk}\| < \dfrac{1}{k}$.

Proof. For λ in Λ_n, we construct $\phi^n_{i,i+1,k}$ in the form

$$\phi^n_{i,i+1,k}(\lambda) = \begin{cases} \phi^n_{i,i+1}(\lambda), & \text{if } |\phi^n_{i,i+1}(\lambda)| \geq \dfrac{1}{kn}; \\ \dfrac{\phi^n_{i,i+1}(\lambda)}{kn|\phi^n_{i,i+1}(\lambda)|}, & \text{if } 0 < |\phi^n_{i,i+1}(\lambda)| < \dfrac{1}{kn}; \\ \dfrac{1}{kn}, & \text{if } \phi^n_{i,i+1}(\lambda) = 0. \end{cases}$$

Thus $\|M_{\phi^n_{i,i+1}} - M_{\phi^n_{i,i+1,k}}\| < \dfrac{1}{k(n-1)}, \quad 1 \leq i \leq n-1$. Therefore we obtain

$$\|A_n - A_{nk}\| \leq \sum_{i=1}^{n-1} \|M_{\phi^n_{i,i+1}} - M_{\phi^n_{i,i+1,k}}\| < (n-1)\frac{1}{k(n-1)} = \frac{1}{k}.$$

By the definition, the operator $M_{\phi^n_{i,i+1,k}}$ is invertible for $1 \leq i \leq n-1$. ∎

LEMMA 2.4. *If an operator A_n is a direct integral of strongly irreducible operators stated as in (2) in the form*

$$A_n = \begin{pmatrix} M_{\phi_n} & M_{\phi_{12}^n} & M_{\phi_{13}^n} & \cdots & M_{\phi_{1n}^n} \\ 0 & M_{\phi_n} & M_{\phi_{23}^n} & \cdots & M_{\phi_{2n}^n} \\ 0 & 0 & M_{\phi_n} & \cdots & M_{\phi_{3n}^n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M_{\phi_n} \end{pmatrix}_{n\times n}$$

such that $M_{\phi_{i,i+1}^n}$ is invertible in $\mathscr{L}(L^2(\mu_n))$ for $i = 1,2,\ldots,n-1$, then there exists an invertible operator X_n in $\mathscr{L}((L^2(\mu_n))^{(n)})$ such that $X_n A_n X_n^{-1}$ is in the form

$$X_n A_n X_n^{-1} = \begin{pmatrix} M_{\phi_n} & I & 0 & \cdots & 0 \\ 0 & M_{\phi_n} & I & \cdots & 0 \\ 0 & 0 & M_{\phi_n} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M_{\phi_n} \end{pmatrix}. \tag{3}$$

Proof. We construct an invertible upper triangular operator-valued matrix X_n in $\mathscr{L}((L^2(\mu_n))^{(n)})$ as follows.

Choose an invertible operator $M_{f_{nn}}$ in $\mathscr{L}(L^2(\mu_n))$. Fix an operator $M_{f_{ii}}$ by the equation

$$M_{\phi_{i,i+1}^n} M_{f_{i+1,i+1}} = M_{f_{ii}}$$

for $i = 1,\ldots,n-1$. Let $\{M_{f_{ii}}\}_{i=1}^n$ be the main diagonal (0-diagonal) entries of X_n. Notice that every operator in the set $\{M_{f_{ii}}\}_{i=1}^n$ is invertible in $\mathscr{L}(L^2(\mu_n))$.

Choose an operator $M_{f_{n-1,n}^n}$ in $\mathscr{L}(L^2(\mu_n))$. Fix an operator $M_{f_{i,i+1}^n}$ by the equation

$$M_{\phi_{i,i+1}^n} M_{f_{i+1,i+2}^n} + M_{\phi_{i,i+2}^n} M_{f_{i+2,i+2}^n} = M_{f_{i,i+1}^n}$$

for $i = 1,\ldots,n-2$. Let $\{M_{f_{i,i+1}^n}\}_{i=1}^{n-1}$ be the 1-diagonal entries of X_n.

Choose an operator $M_{f_{n-l,n}^n}$ in $\mathscr{L}(L^2(\mu_n))$, where l is a positive integer such that $1 \le l \le n-1$. Fix an operator $M_{f_{i,i+l}^n}$ by the equation

$$M_{\phi_{i,i+1}^n} M_{f_{i+1,i+l+1}^n} + M_{\phi_{i,i+2}^n} M_{f_{i+2,i+l+1}^n} + \cdots + M_{\phi_{i,i+l+1}^n} M_{f_{i+l+1,i+l+1}^n} = M_{f_{i,i+l}^n}$$

for $i = 1,\ldots,n-l-1$. Let $\{M_{f_{i,i+l}^n}\}_{i=1}^{n-l}$ be the l-diagonal entries of X_n.

Choose an operator $M_{f_{1n}^n}$ in $\mathscr{L}(L^2(\mu_n))$ to be the n-diagonal entry of X_n.

Therefore we obtain an invertible operator-valued matrix X_n in the form

$$X_n = \begin{pmatrix} M_{f_{11}} & M_{f^n_{12}} & M_{f^n_{13}} & \cdots & M_{f^n_{1n}} \\ 0 & M_{f_{22}} & M_{f^n_{23}} & \cdots & M_{f^n_{2n}} \\ 0 & 0 & M_{f_{33}} & \cdots & M_{f^n_{3n}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M_{f_{nn}} \end{pmatrix}$$

such that the equation (3) holds. ∎

By Lemma 2.4, we can reduce equation (2) to the form

$$A = \bigoplus_{n=1}^{\infty} \int_{\Lambda_n} \begin{pmatrix} M_{\phi_n} & I & 0 & \cdots & 0 \\ 0 & M_{\phi_n} & I & \cdots & 0 \\ 0 & 0 & M_{\phi_n} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M_{\phi_n} \end{pmatrix}_{n\times n} (\lambda) d\mu_n(\lambda), \tag{4}$$

in the sense of similar equivalence.

For a regular Borel measure ν on $\mathbb{C}$ with compact support K, define N_ν on $L^2(\nu)$ by $N_\nu f = z \cdot f$ for each f in $L^2(\nu)$.

LEMMA 2.5. *Let an operator A_n be in the form*

$$A_n = \begin{pmatrix} N^{(\infty)}_{\nu_n} & I & 0 & \cdots & 0 \\ 0 & N^{(\infty)}_{\nu_n} & I & \cdots & 0 \\ 0 & 0 & N^{(\infty)}_{\nu_n} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N^{(\infty)}_{\nu_n} \end{pmatrix}_{n\times n},$$

where ν_n is a regular Borel measure and supported on some compact set K_n such that $0 < \nu_n(K_n) < \infty$. Then the strongly irreducible decomposition of A_n is not unique up to similarity.

Proof. To prove this lemma, we need to construct two bounded maximal abelian sets of idempotents in $\{A_n\}'$ which are not similar.

We can write $N^{(\infty)}_{\nu_n}$ in the form $N_{\nu_n} \otimes I_{l^2}$, where I_{l^2} is the identity operator on l^2. Denote by $\mathscr{P}$ the set of all the spectral projections of N_{ν_n}. This set forms a bounded maximal abelian set of idempotents in $\{N_{\nu_n}\}'$. Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for l^2. Denote by E_k the projection such that $\operatorname{ran} E_k = \{\lambda e_k : \lambda \in \mathbb{C}\}$. Let $\mathscr{Q}_1 \triangleq \{P \in \mathscr{L}(l^2) : P = P^* = P^2 \in \{E_k : k \in \mathbb{N}\}''\}$. Denote by χ_S the characteristic function for a Borel subset S in the interval $[0,1]$ and let $\hat{\mathscr{Q}}_2 \triangleq \{M_{\chi_S} \in \mathscr{L}(L^2[0,1]) : S \subset [0,1] \text{ is Borel.}\}$. There is a unitary operator $U : L^2[0,1] \to l^2$ such that $UPU^* \in \mathscr{L}(l^2)$ for every $P \in \hat{\mathscr{Q}}_2$. The sets $\mathscr{Q}_2 \triangleq U\hat{\mathscr{Q}}_2U^*$ and $\mathscr{Q}_1$ are two bounded maximal abelian sets of idempotents in

$\mathscr{L}(l^2)$ but they are not unitarily equivalent. By the fact that $W^*(\mathscr{P}) \otimes W^*(\mathscr{Q}_1)$ and $W^*(\mathscr{P}) \otimes W^*(\mathscr{Q}_2)$ are both maximal abelian von Neumann algebras, we obtain that

$$\mathscr{F}_1 \triangleq \{P \in W^*(\mathscr{P}) \otimes W^*(\mathscr{Q}_1) : P = P^* = P^2\}$$

and

$$\mathscr{F}_2 \triangleq \{P \in W^*(\mathscr{P}) \otimes W^*(\mathscr{Q}_2) : P = P^* = P^2\}$$

are both maximal abelian sets of idempotents in $\{N_{\nu_n} \otimes I_{l^2}\}' = L^\infty(\nu_n) \otimes \mathscr{L}(l^2)$.

We need to prove that $\mathscr{F}_i^{(n)}$ is a bounded maximal abelian set of idempotents in $\{A_n\}'$ for $i = 1, 2$.

An operator X in $\{A_n\}'$ can be expressed in the form

$$X = \begin{pmatrix} X_{11} & X_{12} & X_{13} & \cdots & X_{1n} \\ X_{21} & X_{22} & X_{23} & \cdots & X_{2n} \\ X_{31} & X_{32} & X_{33} & \cdots & X_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & X_{n3} & \cdots & X_{nn} \end{pmatrix}_{n\times n} . \tag{5}$$

We prove that X_{ij} is in $\{N_{\nu_n} \otimes I_{l^2}\}'$. Note that $\mathscr{P}^{(\infty)}$ is the set of all the spectral projections of $N_{\nu_n} \otimes I_{l^2}$. Fix an projection P in $(\mathscr{P}^{(\infty)})^{(n)}$. The operator A can be expressed in the form

$$A_n = A_{n1} \oplus A_{n2},$$

where

$$A_{ni} = \begin{pmatrix} N_{\nu_{ni}}^{(\infty)} & I & 0 & \cdots & 0 \\ 0 & N_{\nu_{ni}}^{(\infty)} & I & \cdots & 0 \\ 0 & 0 & N_{\nu_{ni}}^{(\infty)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N_{\nu_{ni}}^{(\infty)} \end{pmatrix}, \ i = 1, 2.$$

The measures ν_{n1} and ν_{n2} are mutually singular and their supports depend on the characteristic functions corresponding to P and $I - P$. Hence X can also be expressed in the form

$$X = \begin{array}{cc} \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} & \begin{array}{l} \operatorname{ran} P \\ \operatorname{ran}(I - P) \end{array} \end{array} .$$

The equations $A_{n1}Y_{12} = Y_{12}A_{n2}$ and $A_{n2}Y_{21} = Y_{21}A_{n1}$ yield that $Y_{12} = Y_{21} = 0$. Therefore P reduces X and X_{ij}s are in $\{N_{\nu_n} \otimes I_{l^2}\}'$. A computation shows that the equation $X_{ij} = 0$ holds for $i > j$ and $X_{ii} = X_{11}$ for $i = 2, \ldots, n$ in (5). Furthermore, if X as in (5) is an idempotent, then so is every main diagonal entry X_{ii} of X.

We assume that X is an idempotent in $\{A_n\}'$ and commutes with $\mathscr{F}_1^{(n)}$. Hence X_{ii} commutes with $\mathscr{F}_1$. The fact that $\mathscr{F}_1$ is a maximal abelian set of idempotents implies that X_{ii} belongs to $\mathscr{F}_1$. Thus X_{ii} commutes with X_{ij}. For the 1-diagonal entries, the equation $2X_{ii}X_{i,i+1} - X_{i,i+1} = 0$ yields $X_{i,i+1} = 0$,

for $i = 1, \dots, n-1$. By this way, the k-diagonal entries of X are all zero, for $k = 2, \dots, n$. Therefore X is in $\mathscr{F}_1^{(n)}$. Both $\mathscr{F}_1^{(n)}$ and $\mathscr{F}_2^{(n)}$ are bounded maximal abelian sets of idempotents in $\{A_n\}'$.

We prove that $\mathscr{F}_1^{(n)}$ and $\mathscr{F}_2^{(n)}$ are not similar in $\{A_n\}'$. Every operator X in $\{A_n\}'$ can be written in the form

$$X = \int_{\sigma(N_{\nu_n})} X(\lambda) d\nu_n(\lambda).$$

Suppose that there is an invertible operator X in $\{A_n\}'$ such that

$$X\mathscr{F}_2^{(n)}X^{-1} = \mathscr{F}_1^{(n)}.$$

For each P in $\mathscr{F}_2^{(n)}$, the projection $P(\lambda)$ is either of rank ∞ or 0, for almost every λ in $\sigma(N_{\nu_n})$. But there exists an projection Q in $\mathscr{F}_1^{(n)}$ such that $Q(\lambda)$ is of rank n, for almost every λ in $\sigma(N_{\nu_n})$. This is a contradiction. Therefore $\mathscr{F}_1^{(n)}$ and $\mathscr{F}_2^{(n)}$ are not similar in $\{A_n\}'$. ∎

By ([22], Theorem 3.3), we have the following corollary.

COROLLARY 2.6. *Let an operator A_n be in the form*

$$A_n = \begin{pmatrix} N_{\nu_n}^{(m)} & I & 0 & \cdots & 0 \\ 0 & N_{\nu_n}^{(m)} & I & \cdots & 0 \\ 0 & 0 & N_{\nu_n}^{(m)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N_{\nu_n}^{(m)} \end{pmatrix}_{n\times n},$$

where m is a positive integer and ν_n is a regular Borel measure supported on some compact set K_n such that $0 < \nu_n(K_n) < \infty$. Then the strongly irreducible decomposition of A_n is unique up to similarity.

For a regular Borel measure ν with compact support, Denote by $J_m(\nu)$ an operator in the form

$$J_m(\nu) = \begin{pmatrix} N_\nu & I & 0 & \cdots & 0 \\ 0 & N_\nu & I & \cdots & 0 \\ 0 & 0 & N_\nu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N_\nu \end{pmatrix}_{m\times m} \begin{matrix} L^2(\nu) \\ L^2(\nu) \\ L^2(\nu) \\ \vdots \\ L^2(\nu) \end{matrix}. \qquad (6)$$

LEMMA 2.7. *Every operator in $\{J_m(\nu)\}'$ is in the form*

$$\begin{pmatrix} M_{\phi_1} & M_{\phi_2} & \cdots & M_{\phi_m} \\ 0 & M_{\phi_1} & \cdots & M_{\phi_{m-1}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{\phi_1} \end{pmatrix},$$

where ϕ_i is in $L^\infty(\nu)$ for $1 \leq i \leq m$.

Proof. By a similar computation as in Lemma 2.5, we obtain that every operator in $\{J_m(\nu)\}'$ is in the form

$$\begin{pmatrix} M_{\phi_{11}} & M_{\phi_{12}} & \cdots & M_{\phi_{1m}} \\ 0 & M_{\phi_{22}} & \cdots & M_{\phi_{2,m-1}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{\phi_{mm}} \end{pmatrix}.$$

By the equation

$$N_\nu M_{\phi_{i,j+1}} + M_{\phi_{i+1,j+1}} = M_{\phi_{i,j}} + M_{\phi_{i,j+1}} N_\nu,$$

the k-diagonal entries are as required for $1 \leq k \leq n$. ∎

LEMMA 2.8. *Let m_1 and m_2 be two positive integers such that $m_1 > m_2$. Then the following equations hold:*

(i) $\{B \in \mathscr{L}((L^2(\nu))^{(m_2)}, (L^2(\nu))^{(m_1)}) : J_{m_1}(\nu)B = BJ_{m_2}(\nu)\}$
$= \{(C^T, 0)^T : C \in \{J_{m_2}(\nu)\}'\}$.

(ii) $\{B \in \mathscr{L}((L^2(\nu))^{(m_1)}, (L^2(\nu))^{(m_2)}) : J_{m_2}(\nu)B = BJ_{m_1}(\nu)\}$
$= \{(0, C) : C \in \{J_{m_2}(\nu)\}'\}$.

Proof. We only need to prove the first equation. The second equation can be obtained by the same method. Let $B = (C^T, D^T)^T$ such that C and D are in the form

$$C = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1m_2} \\ B_{21} & B_{22} & \cdots & B_{2m_2} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m_2 1} & B_{m_2 2} & \cdots & B_{m_2 m_2} \end{pmatrix}, \quad D = \begin{pmatrix} B_{m_2+1,1} & \cdots & B_{m_2+1,m_2} \\ \vdots & & \vdots \\ B_{m_1 1} & \cdots & B_{m_1 m_2} \end{pmatrix}.$$

By a similar computation as in Lemma 2.5, we can obtain that $P^{(m_1)}B = BP^{(m_2)}$ for every spectral projection P of N_ν. Thus, every B_{ij} belongs to $\{N_\nu\}'$, for $1 \leq i \leq m_1$ and $1 \leq j \leq m_2$. For $i = 1, \ldots, m_1 - 1$, the equation $N_\nu B_{i1} + B_{i+1,1} = B_{i1}N_\nu$ yields $B_{i+1,1} = 0$. For $i = 2, \ldots, m_1 - 1$, the equation $N_\nu B_{i2} + B_{i+1,2} = B_{i2}N_\nu$ yields $B_{i+1,2} = 0$. By this way, we can obtain $B_{ij} = 0$ for $i > j$. Hence $D = 0$ and a further computation shows that $C \in \{J_{m_2}(\nu)\}'$. ∎

LEMMA 2.9. *Let m_1, m_2, r_1, r_2 be positive integers. If an idempotent P in $M_n(\mathbb{C})$ is in the form*

$$P = \begin{pmatrix} I_{m_1} & 0 & R_{11} & R_{12} \\ 0 & 0_{r_1} & R_{21} & R_{22} \\ 0 & 0 & I_{m_2} & 0 \\ 0 & 0 & 0 & 0_{r_2} \end{pmatrix},$$

where I_m is the identity operator in $M_m(\mathbb{C})$, then there exists an invertible operator X in the form

$$X = \begin{pmatrix} I_{m_1} & 0 & 0 & R_{12} \\ 0 & I_{r_1} & -R_{21} & 0 \\ 0 & 0 & I_{m_2} & 0 \\ 0 & 0 & 0 & I_{r_2} \end{pmatrix} \text{ and } X^{-1} = \begin{pmatrix} I_{m_1} & 0 & 0 & -R_{12} \\ 0 & I_{r_1} & R_{21} & 0 \\ 0 & 0 & I_{m_2} & 0 \\ 0 & 0 & 0 & I_{r_2} \end{pmatrix}$$

such that

$$XPX^{-1} = \begin{pmatrix} I_{m_1} & 0 & 0 & 0 \\ 0 & 0_{r_1} & 0 & 0 \\ 0 & 0 & I_{m_2} & 0 \\ 0 & 0 & 0 & 0_{r_2} \end{pmatrix}.$$

Note that the equation $P^2 = P$ implies that $R_{11} = R_{22} = 0$ and the construction of X depends on P. In the following example, we construct an operator A and prove the strongly irreducible decomposition of A is unique up to similarity.

EXAMPLE 2.10. Let $A = J_3^{(2)}(\nu) \oplus J_2^{(3)}(\nu) \oplus J_1^{(2)}(\nu)$. We prove that for every two bounded maximal abelian sets of idempotents $\mathscr{P}$ and $\mathscr{Q}$ in $\{A\}'$, there is an invertible operator X in $\{A\}'$ such that the equation $\mathscr{P} = X\mathscr{Q}X^{-1}$ holds and

$$V(\{A\}') = \{f \text{ is bounded Borel} : \sigma(N_\nu) \to \mathbb{N} \oplus \mathbb{N} \oplus \mathbb{N}\},$$

$$K_0(\{A\}') = \{f \text{ is bounded Borel} : \sigma(N_\nu) \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}\}.$$

Denote by $\mathscr{P}_{m_1}$ the set of all the idempotents in $\{J_{m_1}(\nu)\}'$. Note that $\mathscr{P}_{m_1}$ equals the set of all the spectral projections of $N_\nu^{(m_1)}$. Denote by $\mathscr{E}_{m_2}$ the set of all the diagonal projections in $M_{m_2}(\mathbb{C})$ and by $\mathscr{F}_{m_1,m_2}$ the set of all the projections in $\{\mathscr{P}_{m_1} \otimes \mathscr{E}_{m_2}\}''$.

Let $\mathscr{P} = \mathscr{F}_{3,2} \oplus \mathscr{F}_{2,3} \oplus \mathscr{F}_{1,2}$. We can verify that $\mathscr{P}$ is a bounded maximal abelian set of idempotents in $\{A\}'$. Then we only need to prove that for every bounded maximal abelian set of idempotents $\mathscr{Q}$ in $\{A\}'$, there is an invertible operator X in $\{A\}'$ such that $\mathscr{P} = X\mathscr{Q}X^{-1}$.

We reduce the rest into two claims:

(i) For every idempotent P in $\{A\}'$, there is an invertible operator X in $\{A\}'$ such that XPX^{-1} belongs to $\mathscr{P}$.

(ii) There are seven idempotents $\{Q_k\}_{k=1}^7$ in $\mathscr{Q}$ such that for almost every λ in $\sigma(N_\nu)$, $\{Q_k(\lambda)\}_{k=1}^7$ and $\mathscr{Q}(\lambda)$ generate the same bounded maximal abelian set of idempotents.

Every operator B in $\{A\}'$ can be expressed in the form

$$B = \begin{pmatrix} B^{11} & B^{12} & \cdots & B^{17} \\ B^{21} & B^{22} & \cdots & B^{27} \\ \vdots & \vdots & \ddots & \vdots \\ B^{71} & B^{72} & \cdots & B^{77} \end{pmatrix},$$

where

$$B^{11} = \begin{pmatrix} b_1^{11} & b_2^{11} & b_3^{11} \\ 0 & b_1^{11} & b_2^{11} \\ 0 & 0 & b_1^{11} \end{pmatrix}, \quad B^{13} = \begin{pmatrix} b_1^{13} & b_2^{13} \\ 0 & b_1^{13} \\ 0 & 0 \end{pmatrix}, \quad B^{16} = \begin{pmatrix} b_1^{16} \\ 0 \\ 0 \end{pmatrix},$$

$$B^{31} = \begin{pmatrix} 0 & b_1^{31} & b_2^{31} \\ 0 & 0 & b_1^{31} \end{pmatrix}, \quad B^{33} = \begin{pmatrix} b_1^{33} & b_2^{33} \\ 0 & b_1^{33} \end{pmatrix}, \quad B^{36} = \begin{pmatrix} b_1^{36} \\ 0 \end{pmatrix},$$

$$B^{61} = \begin{pmatrix} 0 & 0 & b_1^{31} \end{pmatrix}, \quad B^{63} = \begin{pmatrix} 0 & b_1^{33} \end{pmatrix}, \quad B^{66} = \begin{pmatrix} b_1^{66} \end{pmatrix},$$

other B_{ij}s are expressed as follows:

- For $1 \leq i, j \leq 2$, B^{ij}s are of the same form;
- For $3 \leq i, j \leq 5$, B^{ij}s are of the same form;
- For $6 \leq i, j \leq 7$, B^{ij}s are of the same form;
- For $1 \leq i, j \leq 2$ and $3 \leq j \leq 5$, B^{ij}s are of the same form;
- For $1 \leq i, j \leq 2$ and $6 \leq j \leq 7$, B^{ij}s are of the same form;
- For $3 \leq i, j \leq 5$ and $1 \leq j \leq 2$, B^{ij}s are of the same form;
- For $3 \leq i, j \leq 5$ and $6 \leq j \leq 7$, B^{ij}s are of the same form;
- For $6 \leq i, j \leq 7$ and $1 \leq j \leq 2$, B^{ij}s are of the same form;
- For $6 \leq i, j \leq 7$ and $3 \leq j \leq 5$, B^{ij}s are of the same form,

where b_k^{ij}s belong to $\{N_\nu\}'$ for $1 \leq i, j \leq 7$ and $1 \leq k \leq 3$.

For B expressed in the above form, there is a unitary operator U_1 such that

$$U_1 B U_1^* = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ 0 & B_{22} & B_{23} \\ 0 & 0 & B_{33} \end{pmatrix},$$

where B_{ij}s are in the form

$$B_{11} = \begin{pmatrix} b_1^{11} & b_1^{12} & \vdots & b_1^{13} & b_1^{14} & b_1^{15} & \vdots & b_1^{16} & b_1^{17} \\ b_1^{21} & b_1^{22} & \vdots & b_1^{23} & b_1^{24} & b_1^{25} & \vdots & b_1^{26} & b_1^{27} \\ \cdots & \cdots & & \cdots & \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & \vdots & b_1^{33} & b_1^{34} & b_1^{35} & \vdots & b_1^{36} & b_1^{37} \\ 0 & 0 & \vdots & b_1^{43} & b_1^{44} & b_1^{45} & \vdots & b_1^{46} & b_1^{47} \\ 0 & 0 & \vdots & b_1^{53} & b_1^{54} & b_1^{55} & \vdots & b_1^{56} & b_1^{57} \\ \cdots & \cdots & & \cdots & \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & \vdots & 0 & 0 & 0 & \vdots & b_1^{66} & b_1^{67} \\ 0 & 0 & \vdots & 0 & 0 & 0 & \vdots & b_1^{76} & b_1^{77} \end{pmatrix}_{7\times 7}, \tag{7}$$

$$B_{12} = \begin{pmatrix} b_2^{11} & b_2^{12} & \vdots & b_2^{13} & b_2^{14} & b_2^{15} \\ b_2^{21} & b_2^{22} & \vdots & b_2^{23} & b_2^{24} & b_2^{25} \\ \cdots & \cdots & & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & b_2^{33} & b_2^{34} & b_2^{35} \\ 0 & 0 & \vdots & b_2^{43} & b_2^{44} & b_2^{45} \\ 0 & 0 & \vdots & b_2^{53} & b_2^{54} & b_2^{55} \\ \cdots & \cdots & & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & 0 & 0 & 0 \\ 0 & 0 & \vdots & 0 & 0 & 0 \end{pmatrix}_{7\times 5}, \quad B_{13} = \begin{pmatrix} b_3^{11} & b_3^{12} \\ b_3^{21} & b_3^{22} \\ \cdots & \cdots \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \cdots & \cdots \\ 0 & 0 \\ 0 & 0 \end{pmatrix}_{7\times 2},$$

$$B_{22} = \begin{pmatrix} b_1^{11} & b_1^{12} & \vdots & b_1^{13} & b_1^{14} & b_1^{15} \\ b_1^{21} & b_1^{22} & \vdots & b_1^{23} & b_1^{24} & b_1^{25} \\ \cdots & \cdots & & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & b_1^{33} & b_1^{34} & b_1^{35} \\ 0 & 0 & \vdots & b_1^{43} & b_1^{44} & b_1^{45} \\ 0 & 0 & \vdots & b_1^{53} & b_1^{54} & b_1^{55} \end{pmatrix}, \quad B_{23} = \begin{pmatrix} b_2^{11} & b_2^{12} \\ b_2^{21} & b_2^{22} \\ \cdots & \cdots \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$B_{33} = \begin{pmatrix} b_1^{11} & b_1^{12} \\ b_1^{21} & b_1^{22} \end{pmatrix}.$$

If P is an idempotent in $\{U_1AU_1^*\}'$, then by the proof of ([22], Lemma 3.4), we can construct an invertible operator X in $\{U_1AU_1^*\}'$ of the form

$$X=\begin{pmatrix} X_1 & & \\ & X_2 & \\ & & X_3 \end{pmatrix}\oplus\begin{pmatrix} X_1 & \\ & X_2 \end{pmatrix}\oplus X_1,$$

such that

$$XPX^{-1}=\begin{pmatrix} P_{11} & P_{12} & P_{13} \\ 0 & P_{22} & P_{23} \\ 0 & 0 & P_{33} \end{pmatrix},$$

where the main diagonal blocks as in (7) are diagonal projections. There is also a unitary operator U_2 in $\{U_1AU_1^*\}'$ of the form

$$U_2=\begin{pmatrix} U_1^2 & & \\ & U_2^2 & \\ & & U_3^2 \end{pmatrix}\oplus\begin{pmatrix} U_1^2 & \\ & U_2^2 \end{pmatrix}\oplus U_1^2,$$

such that the equation

$$\begin{pmatrix} U_1^2 & & \\ & U_2^2 & \\ & & U_3^2 \end{pmatrix} P_{11} \begin{pmatrix} U_1^2 & & \\ & U_2^2 & \\ & & U_3^2 \end{pmatrix}^* (\lambda)$$

$$=\begin{pmatrix} I_{s_1} & 0 & \vdots & * & * & \vdots & * & * \\ 0 & 0_{t_1} & \vdots & * & * & \vdots & * & * \\ \cdots & \cdots & & \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & \vdots & I_{s_2} & 0 & \vdots & * & * \\ 0 & 0 & \vdots & 0 & 0_{t_2} & \vdots & * & * \\ \cdots & \cdots & & \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & \vdots & 0 & 0 & \vdots & I_{s_3} & 0 \\ 0 & 0 & \vdots & 0 & 0 & \vdots & 0 & 0_{t_3} \end{pmatrix}$$

holds for almost every λ in $\sigma(N_\nu)$, where s_i and t_i are non-negative integers.

Write $U_2XPX^{-1}U_2^*$ in the form

$$\widehat{P}=U_2XPX^{-1}U_2^*=\begin{pmatrix} \widehat{P}_{11} & \widehat{P}_{12} & \widehat{P}_{13} \\ 0 & \widehat{P}_{22} & \widehat{P}_{23} \\ 0 & 0 & \widehat{P}_{33} \end{pmatrix}.$$

By Lemma 2.9, we can construct an invertible operator Y_1 in $\{U_1AU_1^*\}'$ such that $\widehat{P}_{11}$, $\widehat{P}_{22}$, and $\widehat{P}_{33}$ become diagonal projections after similar transformation. Furthermore, we can construct an invertible operator Y_2 in $\{U_1AU_1^*\}'$ such that the 1-diagonal blocks of $Y_1\widehat{P}Y_1^{-1}$ vanish after similar transformation. And then we can construct an invertible operator Y_3 in $\{U_1AU_1^*\}'$ such that the 2-diagonal blocks

of $Y_2Y_1\widehat{P}Y_1^{-1}Y_2^{-1}$ vanish after similar transformation. Thus we finish the proof of claim (i).

To prove claim (ii), we need to define a ν-measurable function r_Q with respect to an idempotent Q in $\{A\}'$. Without loss of generality, we assume that $Q \sim (P_{31} \oplus P_{32}) \oplus (P_{21} \oplus P_{22} \oplus P_{23}) \oplus (P_{11} \oplus P_{12}) \in \mathscr{F}_{3,2} \oplus \mathscr{F}_{2,3} \oplus \mathscr{F}_{1,2}$. Define

$$\begin{aligned} r_Q(\lambda) \quad \triangleq \quad & \frac{1}{3}\mathrm{Tr}(P_{31}(\lambda) + P_{32}(\lambda)) \\ & + \frac{1}{2}\mathrm{Tr}(P_{21}(\lambda) + P_{22}(\lambda) + P_{22}(\lambda)) \\ & + \ \mathrm{Tr}(P_{11}(\lambda) + P_{12}(\lambda)),\ \lambda \in \sigma(N_\nu), \end{aligned}$$

where Tr stands for the standard trace of a square matrix. Note that r_Q stays invariant up to similarity. By the proof of ([22], Lemma 3.5), we can obtain that there are seven idempotents Q_i in $\mathscr{Q}$ such that

- the equation $r_{Q_i}(\lambda) = 1$ holds a. e. on $\sigma(N_\nu)$ for $i = 1, \dots, 7$ and
- the equation $Q_iQ_j = 0$ holds for $i \neq j$.

The idempotent Q_i may be in the form

$$Q_i(\lambda) \sim \begin{cases} I_3 \oplus 0, & \lambda \in \Lambda_3; \\ I_2 \oplus 0, & \lambda \in \Lambda_2; \\ I_1 \oplus 0, & \lambda \in \Lambda_1, \end{cases}$$

where $\{\Lambda_i\}_{i=1}^3$ is a Borel partition of $\sigma(N_\nu)$. We can choose finitely many spectral projections of $N_\nu^{(7)}$ to cut Q_is and to piece together new Q_is such that every Q_i belongs to a $\mathscr{P}_m$ for $m = 1, 2, 3$. We finish the proof of claim (ii).

By the proof of ([22], Lemma 3.6) and the idempotents $\{Q_i\}_{i=1}^7$ constructed above, we can obtain an invertible operator X in $\{A\}'$ such that $X\mathscr{Q}X^{-1} = \mathscr{P}$. Therefore the strongly irreducible decomposition of A is unique up to similarity.

Assume that Q_1, Q_2 and Q_3 are in $\mathscr{P}_1$, $\mathscr{P}_2$ and $\mathscr{P}_3$ respectively. Then there is a group isomorphism α such that $\alpha([Q_1]) = (1,0,0)$, $\alpha([Q_2]) = (0,1,0)$, and $\alpha([Q_3]) = (0,0,1)$, where $[Q_i]$ stands for the similar equivalence class of Q_i in $V(\{A\}') = \bigcup_{n=1}^\infty M_n(\{A\}')/\sim$. Thus we obtain $\alpha([I]) = (2,3,2)$, where I is the identity operator in $\{A\}'$. Furthermore, a routine computation yields that $V(\{A\}')$ and $K_0(\{A\}')$ are of the forms at the beginning of this example.

Proof of Theorem 1.1. By the calculating in the above example, we can prove that the strongly irreducible decomposition of $\bigoplus_{i=1}^k J_{m_i}^{(n_i)}(\nu)$ is unique up to similarity, where m_i, n_i and k are all positive integers.

There is a unitary operator V such that VAV^* can be expressed in the form as described at the beginning of Example 2.10. Then we can apply the above lemmas to perform calculation as we need. Note that the equation

$$\{(\bigoplus_{i=1}^{k_1} J_{m_i}^{(n_i)}(\nu_1)) \oplus (\bigoplus_{j=1}^{k_2} J_{m_j}^{(n_j)}(\nu_2))\}' = \{\bigoplus_{i=1}^{k_1} J_{m_i}^{(n_i)}(\nu_1)\}' \oplus \{\bigoplus_{j=1}^{k_2} J_{m_j}^{(n_j)}(\nu_2)\}'$$

holds for mutually singular Borel measures ν_1 and ν_2.

By Lemma 2.5, if the strongly irreducible decomposition of A is unique up to similarity, then every multiplicity function m_{ϕ_n} is bounded. Then we can obtain that $V(\{A\}')$ and $K_0(\{A\}')$ are as described in the theorem. On the other hand, if the strongly irreducible decomposition of A is not unique up to similarity, then there is a number m in $\{n \in \mathbb{N} : \mu(\Lambda_n) > 0\}$ such that the multiplicity function m_{ϕ_m} takes ∞ in its codomain on a Borel subset Γ_{m1} of measure nonzero in its domain. Therefore in $K_0(\{A\}')$, every Borel function f vanishes on Γ_{m1}. This is a contradiction. ∎

The proof of Theorem 1.2 is an application of Lemma 2.3.

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